HARMONIC TRANSFORMATIONS THEOREMS: LINEAR AND INVERSION EQUIVALENCES OF NEO-RIEMANNIAN FUNCTIONS IN SIMPLICIAL COMPLEXES

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Abstract. The Neo-Riemannian theory uses the composition of transformations PLR. The functions PLR^{Ψ} can be seen as a triple composition of all these functions. In this article, we demonstrate a refinement of the calculation of the equivalences of PLR^{Ψ}, PLR and I_n with a single element of a triad in root position, through twelve theorems. Then we build a particular form of matrix expressions for each PLR^{Ψ}, which were proved in six theorems. In this sense, the possibility of finding new transposition or inversion symmetries through matrix algebra is underlined. From these particular propositions, twelve "Harmonic Transformations Theorems" are demonstrated to quickly calculate the PLR,PLR^{Ψ} and inversion I_n equivalences on any simplicial complex $\mathcal{K}^{u}_{TI}[\delta_1, \delta_2, \delta_3]$ of chromatic and diatonic systems.
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1. Introduction

One advantage of multiple representation of the same object is to identify new properties and implicit connections. Musical chords and harmonic progressions has been represented in different geometric forms. Tonnetz represents chords through the infinite tessellation of equilateral triangles; using primarily P, R, L functions to describe a progression. Neo-Riemannian theory and its many studies have analyzed concert and popular music through this transformational methodology [Lew87], [Hye95], [Coh97], [Coh98], [Cap04], [GR11]. Chords can also be represented on a circumference. In this space, the rotation and reflection symmetries are mainly observed. In this line we have works of [For73], [Clo98], [Tym09], [Sch08].

Another representation that enables new connections and calculations are linear structures. Fiore and Noll [FN18] reviewed the equivalences of the Neo-Riemannian functions with the inversion I_n functions. Consequently, they constructed three equivalent linear representations for these usual PLR functions. They also find that the equivalencies of these matrices are determined by the triad in use. If the input chord is not in the closed root position; that is, in the conventional form of root, third, and fifth; the matrix represents a different PLR function [FN18, 14-15]. Hence, they generalize these six combinatorial characteristics of triad positions using group theory and linear representations.

The group context gives rise to the composition of the functions PLR. Some progressions between consecutive chords do not share two notes in common. A special type of composition that includes the three simple Neo-Riemannian functions simultaneously and that we will denote with the superscript Ψ are the PLR^{Ψ} functions. The function P^{Ψ} was studied by Lewin [Lew87], the function L^{Ψ} is used by Cohn [Coh98]. Morris [Mor98] also adds a function R^{Ψ}. In figures 1,2, and 3 it can be seen how operate P^{Ψ} (yellow), L^{Ψ} (gray), R^{Ψ} (red); in three different spaces: Tonnetz, Circumference of pitch classes and Vector Space.

More recently, Jedrzejewski[Jed19] creates a JQZ group analogous to the PLR group. This new group allows calculating inversion equivalences with just one base element of a major or minor triad. Also, JQZ can be extended to atonal triads, where the construction does not obey overlapping by third intervals. The JQZ group also tests its power as an analytical tool to find more balanced symmetries in commutative diagrams than the PLR group.

Furthermore, the PLR functions and their compositions can be generalized to different simplicial complexes spaces. Conventional Tonnetz [Coh97] belongs to the skeletal $\mathcal{S}_1$ display of the simplicial complex $\mathcal{K}^{u}_{T}I3,4,5$. Bigo and Andreatta [BA16] show that the traces of the harmonic progressions can be submerged in other spaces with different interval structures. So, PLR functions keep the same trace in other spaces but represent other types of chords at the harmonic level. In HexaChord [Big13] you can

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do dives and transformations of the progressions and save the transformations in MIDI format, but for the manual analytical task it is difficult, for example, to find the equivalent PLR calculations and the inversions I_n in the circumference.

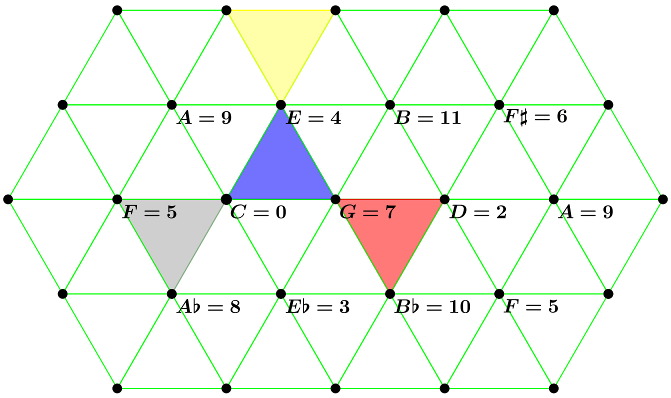


FIGURE 1. PLR^{Ψ} functions over Tonnetz. P^{Ψ} in yellow, R^{Ψ} in red, L^{Ψ} in gray. The major base triad is plotted on the central blue equilateral triangle.

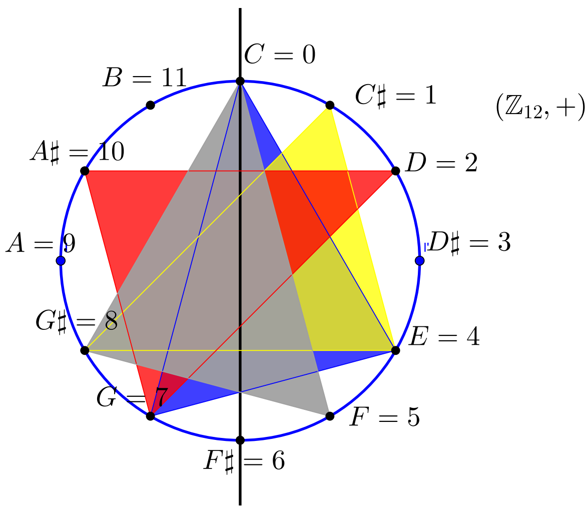


FIGURE 2. PLR^{Ψ} functions over circumference. P^{Ψ} in yellow, R^{Ψ} in red, L^{Ψ} in gray. The major base triad is graphed in blue, behind all triads.

The objective of this article focuses on equivalent transformations with closed root positions [FN18, 15] of the triad chord. First, we will demonstrate equivalence formulas for the functions PLR^{Ψ} studied in Morris [Mor98] with the inversions I_n of the group T/I, refining the calculation to a single operation with a single element of the chord. Second, the contextual matrix representations of the functions PLR^{Ψ} of triads in closed fundamental root position and output in closed root position are constructed. Then, with each matrix theorem a corollary is deduced that expresses the matrix equivalence between the inversions I_n and functions PLR^{Ψ}, only by permutation of the associated matrix. Similarly, a particular way of calculating PLR equivalencies is proposed using only the third of the triad chord. In this sense, we elaborate a comparative table for calculating inversion equivalences with $PLR, PLR\Psi$ and group JQZ of [Jed19]. Finally, these results will help us to build general equivalence formulas in any chromatic and diatonic simplicial space [Big13], which relate the function PLR, $PLR\Psi$ in the Tonnetz, the inversion I_n in the circumference and its matrix representation of vectors.

In summary, the purpose of these formal constructions can be shown as a bidirectional transit between a major triads and minor triads in different spaces through the following diagram:

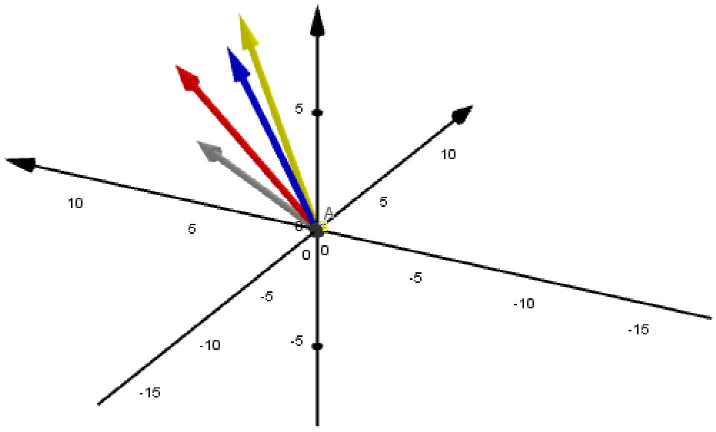
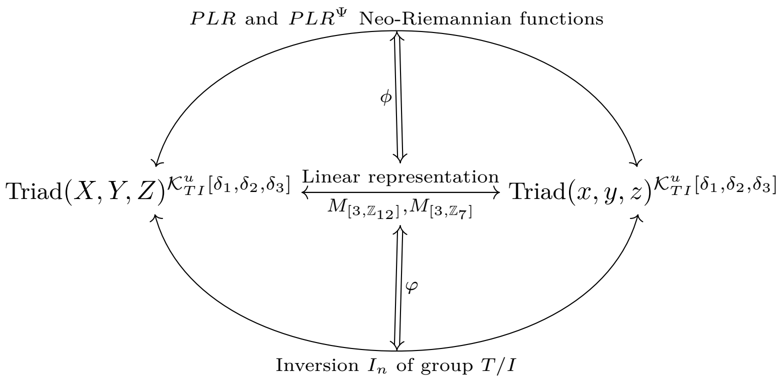


FIGURE 3. PLR^{Ψ} functions over vector space. P^{Ψ} in yellow, R^{Ψ} in red, L^{Ψ} in gray. Major triad base is plotted on the blue vector.



2. Inversion Equivalences for PLR^{Ψ} functions

Since we will work with major and minor triad chords as defined below, we will use the X, Y, Z notation in uppercase for major triads and x, y, z in lowercase for minor triads. These definitions, explained by the variable w, will serve to prove the theorems.

Definition 2.1 (Major Triad Chord)**.**

$$(X, Y, Z) = (w, w + 4, w + 7)$$

Definition 2.2 (Minor Triad Chord)**.**

$$(x, y, z) = (w, w + 3, w + 7)$$

The PLR^{Ψ} functions described in [Lew87], [Coh98], and [Mor98] can be considered analogous to the simple PLR functions. The transformations PLR^{Ψ} from one triad in root position to another triad in same position are defined below as well:

Definition 2.3 (Analogous Parallel P^{Ψ} for major triad)**.**

$$P^{\Psi}(X, Y, Z) = (X + 1, Y, Z + 1)$$

Definition 2.4 (Analogous Parallel P^{Ψ} for minor triad)**.**

$$P^{\Psi}(x,y,z) = (x-1, y, z-1)$$

Definition 2.5 (Analogous Relative R^{Ψ} for major triad)**.**

$$R^{\Psi}(X,Y,Z) = (Z, X-2, Y-2)$$

Definition 2.6 (Analogous Relative R^{Ψ} for minor triad)**.**

$$R^{\Psi}(x,y,z) = (y+2, z+2, x)$$

Definition 2.7 (Analogous Leading-tone exchange L^{Ψ} for major triad)**.**

$$L^{\Psi}(X,Y,Z) = (Y+1, Z+1, X)$$

Definition 2.8 (Analogous Leading-tone exchange L^{Ψ} for minor triad)**.**

$$L^{\Psi}(x,y,z) = (z, x-1, y-1)$$

Finally, the inversion function is geometrically an axial reflection plus a determined rotation. When applied, it transforms each element of the triad into its negative inverse and adds a transposition to each one as follows:

Definition 2.9 (Inversion function for major triads)**.**

$$I_n(X,Y,Z) = (-X+n, -Y+n, -Z+n)$$

Definition 2.10 (Inversion function for minor triads)**.**

$$I_n(x,y,n) = (-x+n, -y+n, -z+n)$$

Amiot [Ami17] points out that the equivalences of the functions PLR with the inversions I_n operate contextually and Jedrzejewski [Jed19, 150] observed that are specifically determinated to the position of the triad. In this section we demonstrate the calculation of contextual PLR^{Ψ} and PLR functions for when a chord is transformed from root position (fundamental, third and fifth) into a chord in descending root position, as performed by the inversion I_n in the definitions. The following theorems refine the calculation of contextual equivalences between PLR^{Ψ}, PLR and I_n demonstrated by [DV20] where the sum of two elements of the chord plus a constant positive or negative was used, for example $P^{\Psi}(X,Y,Z) = I_{X+Z+1}(X,Y,Z)$. And it is synthesized in the multiplication by 2 of the invariant element in the transformation. This means that these addition operations are reduced to multiplying by 2 a single specific element of the triad chord: 2 × root, 2 × third, or 2 × fifth of the chord.

Theorem 2.11. *For every major triad (X,Y,Z) it is complied that the function P^{Ψ} is equal to the inversion function I_n, such that n is the double of the third of the triad. This is expressed through the formula:*

$$P^{\Psi}(X,Y,Z) := I_{2Y}(X,Y,Z) \tag{1}$$

Proof. By definition 2.1, $(X,Y,Z) = (w, w+4, w+7)$. Inversion I transform (X,Y,Z) into $(-X,-Y,-Z)$. So, major triad inverted is expressed $[-(w), -(w+4), -(w+7)]$. On the other hand, we also derive that $2Y = 2(w+4)$. Then, the equation hypothesis affirms that adding $2w+8$ to each inverted element of major triad is equal to the descending minor triad $(Z+1, Y, X+1) = [(w+7)+1, (w+4), (w)+1]$ of P^{ψ} function over (X,Y,Z). So, $[-w+(2w+8), -w-4+(2w+8), -w-7+(2w+8)] = (w+8, w+4, w+1)$. This completes the proof and verifies the formula. ☐

Theorem 2.12. *For every minor triad (x,y,z) it is complied that the function P^{Ψ} is equal to the inversion function I_n, such that n is the double of the third of the triad. This is expressed through the formula:*

$$P^{\Psi}(x,y,z) := I_{2y}(x,y,z) \tag{2}$$

Proof. By definition 2.2, $(x,y,z) = (w, w+3, w+7)$. Inversion I transform (x,y,z) into $(-x,-y,-z)$. So, minor triad inverted is expressed $[-(w), -(w+3), -(w+7)]$. On the other hand, we also derive that $2y = 2(w+3)$. Then, the equation hypothesis affirms that adding $2w+6$ to each inverted element of minor triad is equal to the descending major triad $(z-1, y, x-1) = [(w+7)-1, (w+3), (w)-1]$ of P^{ψ} function over (x,y,z). So, $[-w+(2w+6), -w-3+(2w+6), -w-7+(2w+6)] = (w+6, w+3, w-1)$. This completes the proof and verifies the formula. ☐

Theorem 2.13. *For every major triad (X, Y, Z) it is complied that the function R^Ψ is equal to the inversion function I_n, such that n is the double of the fifth of the triad. This is expressed through the formula:*

$$R^\Psi(X, Y, Z) := I_{2Z}(X, Y, Z) \tag{3}$$

Proof. By definition 2.1, $(X, Y, Z) = (w, w+4, w+7)$. Inversion I transform (X, Y, Z) into $(-X, -Y, -Z)$. So, major triad inverted is expressed $[-(w), -(w + 4), -(w + 7)]$. On the other hand, we also that $2Z = 2(w + 7)$. Then, the equation hypothesis affirms that adding $2w + 14$ to each inverted element of major triad is equal to the descending minor triad $(Y - 2, X - 2, Z) = [(w+4) - 2, (w) - 2, (w+7)]$ of R^ψ function over (X, Y, Z). So, $[-w+(2w+14), -w-4+(2w+14), -w-7+(2w+14)] = (w+2, w-2, w+7)$. This completes the proof and verifies the formula. □

Theorem 2.14. *For every minor triad (x, y, z) it is complied that the function R^Ψ is equal to the inversion function I_n, such that n is the double of the root of triad. This is expressed through the formula:*

$$R^\Psi(x, y, z) := I_{2x}(x, y, z) \tag{4}$$

Proof. By definition 2.2, $(x, y, z) = (w, w + 3, w + 7)$. Inversion I transform (x, y, z) into $(-x, -y, -z)$. So, minor triad inverted is expressed $[-(w), -(w + 3), -(w + 7)]$. On the other hand, we also derive that $2x = 2(w)$. Then, the equation hypothesis affirms that adding $2w$ to each inverted element of minor triad is equal to the descending major triad $(x, z+2, y+2) = [(w), (w+7)+2, (w+3)+2]$ of R^ψ function over (x, y, z). So, $[-w + (2w), -w - 3 + (2w), -w - 7 + (2w)] = (w, w + 9, w + 5)$. This completes the proof and verifies the formula. □

Theorem 2.15. *For every major triad (X, Y, Z) it is complied that the function L^Ψ is equal to the inversion function I_n, such that n is the double of the root of the triad. This is expressed through the formula:*

$$L^\Psi(X, Y, Z) := I_{2X}(X, Y, Z) \tag{5}$$

Proof. By definition 2.1, $(X, Y, Z) = (w, w+4, w+7)$. Inversion I transform (X, Y, Z) into $(-X, -Y, -Z)$. So, major triad inverted is expressed $[-(w), -(w + 4), -(w + 7)]$. On the other hand, we also derive that $2X = 2(w)$. Then, the equation hypothesis affirms that adding $2w$ to each inverted element of major triad is equal to the descending minor triad $(X, Z + 1, Y + 1) = [(w), (w + 7) + 1, (w + 4) + 1]$ of R^ψ function over (X, Y, Z). So, $[-w+(2w), -w-4+(2w), -w-7+(2w)] = (w, w+8, w+5)$. This completes the proof and verifies the formula. □

Theorem 2.16. *For every minor triad (x, y, z) it is complied that the function L^Ψ is equal to the inversion function I_n, such that n is the double of the fifth of the triad. The formula is expressed:*

$$L^\Psi(x, y, z) := I_{2z}(x, y, z) \tag{6}$$

Proof. By definition 2.2, $(x, y, z) = (w, w + 3, w + 7)$. Inversion I transform (x, y, z) into $(-x, -y, -z)$. So, minor triad inverted is expressed $[-(w), -(w + 3), -(w + 7)]$. On the other hand, we also derive that $2z = 2(w + 7)$. Then, the equation hypothesis affirms that adding $2w + 14$ to each inverted element of minor triad is equal to the descending major triad $(y - 1, x - 1, z) = [(w + 3) - 1, (w) - 1, (w + 7)]$ of R^ψ function over (x, y, z). So, $[-w+(2w+14), -w-3+(2w+14), -w-7+(2w+14)] = (w+2, w-1, w+7)$. This completes the proof and verifies the formula. □

3. Linear expressions for PRL^Ψ

The objective is to build a matrix associated with each $PLR\Psi$ function. For this reason, it is important that chords in root position are interpreted as column vectors. Musical objects have been modeled using vectors and matrices. For example, exists exploratory studies of pitch classes and duration of notes as row vectors [Huc89]. Studies that address the difference in action of U, V, W matrices with respect to the 6 closed and open positions of the triad chord [FN18]. More recently, an approach to harmony has been proposed from the formal logic that starts from chords as column vectors. Thus, for example, the *reversion* of a triad [Max18] is similar to the function P^Ψ, that we will study in this section.

In regards to our study, we will adopt the root position notation in [FN18], where the root is the component x, the third is component y and the fifth is component z. The matrix multiplies the triad vector obtaining another triad vector in root position, following the $PLR\Psi$ definitions for each case. As detailed in the introduction, voicings [FN18] are not considered.

4. P^{Ψ} MATRIX AND EQUIVALENT I_n

Theorem 4.1. *The linear representation of P^{Ψ} function for major triads is a 3×3 square matrix with integers coefficients modulo twelve:*

$$P^{\Psi}_{(3,\mathbb{Z}_{12})}(X, Y, Z) := \begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0(X) & + & 2(Y) & + & -1(Z) \\ 0(X) & + & 1(Y) & + & 0(Z) \\ -1(X) & + & 2(Y) & + & 0(Z) \end{bmatrix} = \begin{bmatrix} 2Y - Z \\ Y \\ 2Y - X \end{bmatrix}$$

Then, we have the following equalities using definition 2.3:

$$\begin{array}{lcl} 2Y - Z & = & X + 1 \\ Y & = & Y \\ 2Y - X & = & Z + 1 \end{array}$$

This equality between equations was previously demonstrated in the theorem 2.11 using definition 2.1.

$$\begin{array}{lcl} 2w + 8 - w - 7 & = & w + 1 \\ w + 4 & = & w + 4 \\ 2w + 8 - w & = & w + 7 + 1 \end{array}$$

Structural equality is verified, therefore this concludes the proof. ☐

Corollary 4.1.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix $P^{\Psi}_{(3,\mathbb{Z}_{12})}(X, Y, Z)$,*

$$I_{2Y} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \tag{8}$$

Theorem 4.2. *The linear representation of P^{Ψ} function for minor triads is a 3×3 square matrix with integers coefficients modulo twelve:*

$$P^{\Psi}_{(3,\mathbb{Z}_{12})}(x, y, z) := \begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0(x) & + & 2(y) & + & -1(z) \\ 0(x) & + & 1(y) & + & 0(z) \\ -1(x) & + & 2(y) & + & 0(z) \end{bmatrix} = \begin{bmatrix} 2y - z \\ y \\ 2y - x \end{bmatrix}$$

Then, we have the following equalities using definition 2.4:

$$\begin{array}{lcl} 2y - z & = & x + 1 \\ y & = & y \\ 2y - x & = & z + 1 \end{array}$$

This equality between equations was previously demonstrated in the theorem 2.12 using definition 2.2.

$$\begin{array}{lcl} 2w + 6 - w - 7 & = & w - 1 \\ w + 3 & = & w + 3 \\ 2w + 6 - w & = & w + 7 - 1 \end{array}$$

Structural equality is verified, therefore this concludes the proof. ☐

Corollary 4.2.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix $P^{\Psi}_{(3,\mathbb{Z}_{12})}(x, y, z)$,*

$$I_{2y} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \tag{10}$$

5. R^Ψ matrix and equivalent I_n

Theorem 5.1. *The linear representation of R^Ψ function for major triads is a 3×3 square matrix with integers coefficients modulo twelve:*

$$R^\Psi_{(3,\mathbb{Z}_{12})}(X,Y,Z) := \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{11}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0(X) & + & 0(Y) & + & 1(Z) \\ 0(X) & + & -1(Y) & + & 2(Z) \\ -1(X) & + & 0(Y) & + & 2(Z) \end{bmatrix} = \begin{bmatrix} Z \\ 2Z - Y \\ 2Z - X \end{bmatrix}$$

Then, we have the following equalities using definition 2.5:

$$\begin{aligned} Z &= Z \\ 2Z - Y &= X - 2 \\ 2Z - X &= Y - 2 \end{aligned}$$

This equality between equations was previously demonstrated in the theorem 2.13 using definition 2.1.

$$\begin{aligned} w + 7 &= w + 7 \\ 2w + 14 - w - 4 &= w - 2 \\ 2w + 14 - w &= w + 4 - 2 \end{aligned}$$

Structural equality is verified, therefore this concludes the proof. □

Corollary 5.1.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix $R^\Psi_{(3,\mathbb{Z}_{12})}(X,Y,Z)$,*

$$I_{2Z} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

To find the R^Ψ matrix for minor triads, we apply inverse of R^Ψ for major triads.

Theorem 5.2. *The linear representation of R^Ψ function for minor triads is the inverse of 3×3 square matrix $R^\Psi(X,Y,Z)^{-1}$ with integers coefficients modulo twelve:*

$$[R^\Psi_{(3,\mathbb{Z}_{12})}(X,Y,Z)]^{-1} = R^\Psi_{(3,\mathbb{Z}_{12})}(x,y,z) := \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{13}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2(x) & + & 0(y) & + & -1(z) \\ 2(x) & + & -1(y) & + & 0(z) \\ 1(x) & + & 0(y) & + & 0(z) \end{bmatrix} = \begin{bmatrix} 2x - z \\ 2x - y \\ x \end{bmatrix}$$

Then, we have the following equalities using definition 2.6:

$$\begin{aligned} 2x - z &= y + 2 \\ 2x - y &= z + 2 \\ x &= x \end{aligned}$$

This equality between equations was previously demonstrated in the theorem 2.14 using definition 2.2.

$$\begin{aligned} 2w - w - 7 &= w + 3 + 2 \\ 2w - w - 3 &= w + 7 + 2 \\ w &= w \end{aligned}$$

Structural equality is verified, therefore this concludes the proof. □

Corollary 5.2.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix* $R^{\Psi}_{(3,\mathbb{Z}_{12})}(x,y,z)$,

$$I_{2x} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \tag{14}$$

6. L^{Ψ} matrix and equivalent I_n

Theorem 6.1. *The linear representation of* L^{Ψ} *function for major triads is a* 3×3 *square matrix with integers coefficients modulo twelve:*

$$L^{\Psi}_{(3,\mathbb{Z}_{12})}(X,Y,Z) := \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{15}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2(X) & + & 0(Y) & + & -1(Z) \\ 2(X) & + & -1(Y) & + & 0(Z) \\ 1(X) & + & 0(Y) & + & 0(Z) \end{bmatrix} = \begin{bmatrix} 2X - Z \\ 2X - Y \\ X \end{bmatrix}$$

Then, we have the following equalities using definition 2.7:

$$\begin{array}{ccc} 2X - Z & = & Y + 1 \\ 2X - Y & = & Z + 1 \\ X & = & X \end{array}$$

This equality between equations was previously demonstrated in the theorem 2.15 using definition 2.1.

$$\begin{array}{ccc} 2w - w - 7 & = & w + 4 + 1 \\ 2w - w - 4 & = & w + 7 + 1 \\ w & = & w \end{array}$$

Structural equality is verified, therefore this concludes the proof. ☐

Corollary 6.1.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix* $L^{\Psi}_{(3,\mathbb{Z}_{12})}(X,Y,Z)$,

$$I_{2X} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \tag{16}$$

To find the L^{Ψ} matrix for minor triads, we apply inverse of L^{Ψ} for major triads.

Theorem 6.2. *The linear representation of* L^{Ψ} *function for minor triads is the inverse of* 3×3 *square matrix* $L^{\Psi}(X,Y,Z)^{-1}$ *with integers coefficients modulo twelve:*

$$[L^{\Psi}_{(3,\mathbb{Z}_{12})}(X,Y,Z)]^{-1} = L^{\Psi}_{(3,\mathbb{Z}_{12})}(x,y,z) := \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{17}$$

Proof. By property of matrix product,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0(x) & + & 0(y) & + & 1(z) \\ 0(x) & + & -1(y) & + & 2(z) \\ -1(x) & + & 0(y) & + & 2(z) \end{bmatrix} = \begin{bmatrix} z \\ 2z - y \\ 2z - x \end{bmatrix}$$

Then, we have the following equalities using definition 2.8:

$$\begin{array}{ccc} z & = & z \\ 2z - y & = & x - 1 \\ 2z - x & = & y - 1 \end{array}$$

This equality between equations was previously demonstrated in the theorem 2.16 using definition 2.2.

$$\begin{array}{rcl} w & = & w \\ 2w+14-w-3 & = & w-1 \\ 2w+14-w & = & w+3-1 \end{array}$$

Structural equality is verified, therefore this concludes the proof. ☐

Corollary 6.2.1. *From the previous proof it is obtained the following expression by exchanging first row and third row of matrix* $L^{\Psi}_{(3,\mathbb{Z}_{12})}(x,y,z)$,

$$I_{2z} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

7. Common properties

Neo-Riemmanian P^{Ψ} is an involutive matrix, that satisfies $(P^{\Psi})^2 = I$, and $(P^{\Psi})^{-1} = P^{\Psi}$. However, most interesting properties derived from I_n functions, retaining particular symmetries over R^{Ψ} and L^{Ψ}, as follows:

$$R^{\Psi}_{XYZ} = L^{\Psi}_{xyz} = I_{2Z/2z}$$
$$R^{\Psi}_{xyz} = L^{\Psi}_{XYZ} = I_{2x/2X}$$

$$(R^{\Psi}_{XYZ})^{-1} = (R^{\Psi}_{xyz})$$
$$(L^{\Psi}_{XYZ})^{-1} = (L^{\Psi}_{xyz})$$

8. Another way of calculate PLR and I_n equivalences

This section proposes another particular way to find the equivalence of the Neo-Riemannian PLR functions, using double multiplication of the third of a triad plus a positive or negative constant. The proofs of the propositions indicated here will be summarized and analogous to the previous theorems.

Definition 8.1 (Parallel P function for major triad)**.**

$$P(X,Y,Z) = (X, Y-1, Z)$$

Theorem 8.2. *For every major triad* (X,Y,Z) *it is complied that the function* P *is equal to the inversion function* I_n*, such that* n *is the double of the third of the triad minus 1. This is expressed through the formula:*

$$P(X,Y,Z) := I_{2Y-1}(X,Y,Z) \tag{19}$$

Proof. By definitions 2.1 and 8.1, we need to verify the equation $(-w+2w+8-1, -w-4+2w+8-1, -w-7+2w+8-1) = (w+7, w+3, w)$. So the equation is true for this function. ☐

Definition 8.3 (Parallel P function for minor triad)**.**

$$P(x,y,z) = (x, y+1, z)$$

Theorem 8.4. *For every minor triad* (x,y,z) *it is complied that the function* P *is equal to the inversion function* I_n*, such that* n *is the double of the third of the triad plus 1. This is expressed through the formula:*

$$P(x,y,z) := I_{2y+1}(x,y,z) \tag{20}$$

Proof. By definitions 2.2 and 8.3, we need to verify the equation $(-w+2w+6+1, -w-3+2w+6+1, -w-7+2w+6+1) = (w+7, w+4, w)$. So the equation is true for this function. ☐

Definition 8.5 (Relative R function for major triad)**.**

$$R(X,Y,Z) = (Z+2, X, Y)$$

Theorem 8.6. *For every major triad (X, Y, Z) it is complied that the function R is equal to the inversion function I_n, such that n is the double of the third of the triad minus 4. This is expressed through the formula:*

$$R(X, Y, Z) := I_{2Y-4}(X, Y, Z) \tag{21}$$

Proof. The definitions 2.1 and 8.5 are used. Subsequently, the equation is verified analogously to the previous proofs. □

Definition 8.7 (Relative R function for minor triad)**.**

$$R(x, y, z) = (y, z, x - 2)$$

Theorem 8.8. *For every minor triad (x, y, z) it is complied that the function R is equal to the inversion function I_n, such that n is the double of the third of the triad plus 4. This is expressed through the formula:*

$$R(x, y, z) := I_{2y+4}(x, y, z) \tag{22}$$

Proof. The definitions 2.2 and 8.7 are used. Identically to the theorem 8.6. □

Definition 8.9 (Leading-tone exchange L function for major triad)**.**

$$L(X, Y, Z) = (Y, Z, X - 1)$$

Theorem 8.10. *For every major triad (X, Y, Z) it is complied that the function R is equal to the inversion function I_n, such that n is the double of the third of the triad plus 3. This is expressed through the formula:*

$$L(X, Y, Z) := I_{2Y+3}(X, Y, Z) \tag{23}$$

Proof. The definitions 2.1 and 8.9 are used. Subsequently, the equation is verified analogously to the previous proofs. □

Definition 8.11 (Leading-tone exchange L function for minor triad)**.**

$$L(x, y, z) = (z + 1, x, t)$$

Theorem 8.12. *For every major triad (x, y, z) it is complied that the function R is equal to the inversion function I_n, such that n is the double of the third of the triad minus 3. This is expressed through the formula:*

$$L(x, y, z) := I_{2y-3}(x, y, z) \tag{24}$$

Proof. The definitions 2.2 and 8.11 are used. Identically to the theorem 8.10. □

The theorems of inversion equivalences with simple and compound neo-Riemannian functions up to this section are summarized comparatively in Table 1. The equivalence of the JQZ group is also taken from the table prepared by Jedrzejewski [Jed19, 155]. The particular proposal of this article is based on the calculation of these equivalences as it is usually thought of triads in music: constructions by thirds in root position.

Table 1

PLR group	JQZ group	Jedrzejewski Major	[Jed19, 155] Minor	(X, Y, Z) +Maj	(x, y, z) +Min	(X, Y, Z) × Maj	(x, y, z) ×Min
P	J	I_{2x+7}	I_{2x-7}	I_{X+Z}	I_{x+z}	I_{2Y-1}	I_{2y+1}
R	Z	I_{2x+4}	I_{2x-4}	I_{X+Y}	I_{y+z}	I_{2Y-4}	I_{2y+4}
L	Q	I_{2x+11}	I_{2x-11}	I_{Y+Z}	I_{x+y}	I_{2Y+3}	I_{2y-3}
P^Ψ	QJZ	I_{2x+8}	I_{2x-8}	I_{X+Z+1}	I_{x+z-1}	I_{2Y}	I_{2y}
R^Ψ	JZQ	I_{2x+2}	I_{2x-2}	I_{X+Y-2}	I_{y+z+2}	I_{2Z}	I_{2x}
L^Ψ	JQZ	I_{2x}	I_{2x}	I_{Y+Z+1}	I_{x+y-1}	I_{2X}	I_{2z}

9. PLR, PLR^{Ψ} and Inversión I_n on $\mathcal{K}^u_{TI}[\delta_1, \delta_2, \delta_3]$

The application of simplicial complexes in music theory has several years of use, such as the works of Mazzola [Maz02] and Catanzaro [Cat11], as examples. The Tonnetz can be seen as an $\mathcal{S}$-skeleton of a simplicial complex, thus the traditional Cohn [Coh97] Tonnetz [3,4,5] is an $\mathcal{S}_1$-skeleton, in the same way as can be see the three-dimensional Tonnetz of Gollin [Gol98]. In 2013, Bigo [Big13] presented HexaChord software on Java language. This software of musical analysis allows the visualization of different Tonnetz both chromatic $\mathbb{Z}_{12}$ and diatonics $\mathbb{Z}_7$, the circle of semitones and circle of fifth, with their geometry symmetries. Currently, simplicial chords and their duals have been implemented in online mode in 2020. The objective of this section is to reveal the inversion I_n in the circumference of semitones that corresponds to each neo-Riemannian PLR function living in any of the unfolded support spaces $\mathcal{K}^u_{TI}[\delta_1, \delta_2, \delta_3]$. In this way, by knowing the space y and the chord, the equivalence between the PLR and PLR^{Ψ} function with the inversion can be identified. Later it is shown that the matrix representations $PLR^{\Psi}_{(}3\mathbb{Z}_{12})$ can act perfectly the same in any of the simplicial spaces.

The complex chromatic simplicials that we find in HexaChord are: $\mathcal{K}^u_{TI}[1,1,10]$, $\mathcal{K}^u_{TI}[1,2,9]$, $\mathcal{K}^u_{TI}[1,3,8]$, $\mathcal{K}^u_{TI}[1,4,7]$, $\mathcal{K}^u_{TI}[1,5,6]$, $\mathcal{K}^u_{TI}[2,2,8]$, $\mathcal{K}^u_{TI}[2,3,7]$, $\mathcal{K}^u_{TI}[2,4,6]$, $\mathcal{K}^u_{TI}[2,5,5]$, $\mathcal{K}^u_{TI}[3,3,6]$, $\mathcal{K}^u_{TI}[3,4,5]$ and $\mathcal{K}^u_{TI}[4,4,4]$.

The complex diatonic simplicials that we find in HexaChord are: $\mathcal{K}^u_{TI}[1,1,5]$, $\mathcal{K}^u_{TI}[1,2,4]$, $\mathcal{K}^u_{TI}[1,3,3]$ and $\mathcal{K}^u_{TI}[2,2,3]$.

The following figure will serve to understand the generalized operation of all the support spaces,either a $\mathbb{Z}_{12}$ simplicial space or a $\mathbb{Z}_7$ simplicial space.

According to what is shown in figure 4 the distances between x, y, z correspond to a sum of letters, a, b or c (to make the notation simple). In the usual Tonnetz space $\mathcal{K}^u_{TI}[3,4,5]$, the first distance $a = \delta_1$ corresponds to the first third of the minor chord, wich root is w. Distance $b = \delta_2$ is the other third (major third). Therefore, the sum of the distance a and b is the fifth of the chord. This denoted structure is repeated in all the complexes, which leads us to generalize the following definitions:

Definition 9.1 ("minor chord" cromatic and diatonic generalized)**.**

$$(x, y, z) = (w, w + a, w + a + b)$$

So far you can see three possibilities in the mentioned list of simplicial complexes:

- $a = b$. For example, $\mathcal{K}^u_{TI}[3,3,6]$ and $\mathcal{K}^u_{TI}[2,2,3]$
- $b = c$. For example, $\mathcal{K}^u_{TI}[2,5,5]$ and $\mathcal{K}^u_{TI}[1,3,3]$
- $a = b = c$. For example, $\mathcal{K}^u_{TI}[4,4,4]$

In all cases, the notation for definitions would simply have to be adapted, but the same geometric structure holds. What derives in the same calculations. For $\mathcal{K}^u_{TI}[3,3,6]$, the definition of the minor generalized chord must be $w, w + a, w + 2a$. And it extends in the same way over the other species. For the generalized major chord, the simplicial structure changes position from a to b, $\mathcal{K}^u_{TI}[b, a, c]$.

Definition 9.2 ("major chord" cromatic and diatonic generalized)**.**

$$(X, Y, Z) = (w, w + b, w + a + b)$$

The second point to define are the PLR functions for major and minor triads based on the figure of Tonnetz.

Definition 9.3. *(P functions over minor triad (x,y,z) in K[a,b,c])*

$$P(w, w + a, w + a + b) = (w + a + b, w + b, w)$$

Definition 9.4. *(R function over minor triad (x,y,z) in K[a,b,c])*

$$R(w, w + a, w + a + b) = (w + 2a + b, w + a + b, w + a)$$

Definition 9.5. *(L function over minor triad (x,y,z) in K[a,b,c])*

$$L(w, w + a, w + a + b) = (w + a, w, w - b)$$

Definition 9.6. *(P functions over major triad (X,Y,Z) in K[a,b,c])*

$$P(w, w + b, w + a + b) = (w + a + b, w + a, w)$$

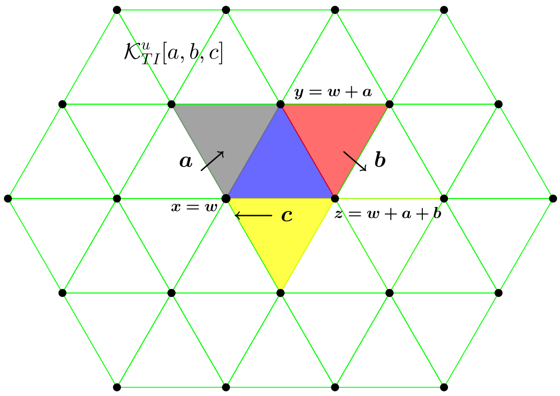


FIGURE 4. PLR functions over simplicial complexes $\mathcal{K}^u_{TI}[a,b,c]$. P in yellow, R in red , L in gray. The major base triad is plotted on the central blue equilateral triangle.

Definition 9.7. *(R functions over major triad (X,Y,Z) in K[a,b,c])*

$$R(w, w+b, w+a+b) = (w+b, w, w-a)$$

Definition 9.8. *(L functions over major triad (X,Y,Z) in K[a,b,c])*

$$L(w, w+b, w+a+b) = (w+2b+a, w+a+b, w+b)$$

Defined the geometric structures, we can proceed to establish generalized theorems of equivalences between Neo-Riemannian functions and the inversion I_n.

Theorem 9.9 (Harmonic Transformation P_1)**.** *The function P in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a+b+c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad plus distances b minus a. This is expressed through the formula:*

$$P^{\mathcal{K}^u_{TI}[a,b,c]}(x,y,z) = I_{2y+b-a}(x,y,z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using the definition 9.1 and 9.3 according to the structure of Tonnetz in figure 4, the generalized P function for "minor chord" in any simplicial space is

$$P(w, w+a, w+a+b) = (w+a+b, w+b, w)$$

. The inversion transform (x,y,z) in $(-w, -w-a, -w-a-b)$. Then, adds $2y+b-a = 2(w+a)+b-a = 2w+2a+b-a = 2w+a+b$ to each inverted element of the triad chord. This should satisfies the structural equality of P function over (x,y,z):

$$\begin{array}{rcc} -w+2w+a+b & = & w+a+b \\ -w-a+2w+a+b & = & w+b \\ -w-a-b+2w+a+b & = & w \end{array}$$

This completes the proof and verifies the formula. □

Theorem 9.10 (Harmonic Transformation R_1)**.** *The function R in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a+b+c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad plus distance b. This is expressed through the formula:*

$$R^{\mathcal{K}^u_{TI}[a,b,c]}(x,y,z) = I_{2y+b}(x,y,z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using the definition 9.1 and 9.4 according to the structure of Tonnetz in figure 4, the generalized R function for "minor chord" in any simplicial space is

$$R(w, w+a, w+a+b) = (w+2a+b, w+a+b, w+a)$$

. The inversion transform (x,y,z) in $(-w, -w-a, -w-a-b)$. Then, adds $2y+b = 2(w+a)+b = 2w+2a+b$ to each inverted element of the triad chord. This should satisfies the structural equality of R function

over (x, y, z):

$$\begin{aligned} -w + 2w + 2a + b &= w + 2a + b \\ -w - a + 2w + 2a + b &= w + a + b \\ -w - a - b + 2w + 2a + b &= w + a \end{aligned}$$

This completes the proof and verifies the formula. □

Musically speaking, we need the root of the "minor triad" to become the fifth of the "major triad". For example, in $\mathcal{K}^u_{TI}[3,4,5]$, R transform $(9,0,4)$ into $(7,4,0)$. The root of the minor triad is $w = 9$. There is another path that leads 9 to 7, adding $a - c$. In this case, the distance c is taken as an inverse vector, as seen in the figure 4. Therefore, $9 + a - c = 7$.

Theorem 9.11 (Harmonic Transformation L_1)**.** *The function l in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad minus distance a. This is expressed through the formula:*

$$L^{\mathcal{K}^u_{TI}[a,b,c]}(x, y, z) = I_{2y-a}(x, y, z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using the definition 9.1 and 9.5 according to the structure of Tonnetz in figure 4, the generalized L function for "minor chord" in any simplicial space is

$$L(w, w + a, w + a + b) = (w + a, w, w - b)$$

. The inversion transform (x, y, z) in $(-w, -w - a, -w - a - b)$. Then, adds $2y - a = 2(w + a) - a = 2w + 2a - a = 2w + a$ to each inverted element of the triad chord. This should satisfies the structural equality of L function over (x, y, z):

$$\begin{aligned} -w + 2w + a &= w + a \\ -w - a + 2w + a &= w \\ -w - a - b + 2w + a &= w - b \end{aligned}$$

This completes the proof and verifies the formula. □

Musically speaking, we need the fifth of the "minor triad" to become the tonic of the "major triad". In the same way as shown with the R function. For example, in $\mathcal{K}^u_{TI}[3,4,5]$, L transform $(4,7,11)$ into $(7,4,0)$. The fifth of the minor triad is $w + a + b = 11$. There is another path that leads 11 to 0, adding $-b + c$. In this case, the distance b is taken as an inverse vector, as seen in the figure 4. Therefore, $11 - b + c = 0$.

Theorem 9.12 (Harmonic Transformation P_2)**.** *The function P in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third Y of the triad plus distances a minus b. This is expressed through the formula:*

$$P^{\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y-b+a}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using definitions 9.2 and 9.6, develop a proof similar to the general theorem for the function P on (x,y,z). □

Theorem 9.13 (Harmonic Transformation R_2)**.** *The function R in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third Y of the triad minus distance b. This is expressed through the formula:*

$$R^{\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y-b}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using definitions 9.2 and 9.7, develop a proof similar to the general theorem for the function R on (x,y,z). □

Theorem 9.14 (Harmonic Transformation L_2)**.** *The function L in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third Y of the triad plus distance a. This is expressed through the formula:*

$$L^{\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y+a}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} \pmod{12}$$

Proof. Using definitions 9.2 and 9.8, develop a proof similar to the general theorem for the function L on (x,y,z). □

Corollary 9.14.1. *The inversion I_n equivalences formulas for the PLR functions on (x, y, z) and $X, Y, Z)$ in diatonic spaces $\mathcal{K}^u_{TI}[a, b, c]$, where $a + b + c \equiv 0 \pmod 7$, works in the same way.*

For linear representation, it can be proven that the Fiore and Noll [FN18] matrices W, V, U, works correctly in any simplicial support space $\mathcal{K}^u_{TI}[a, b, c]$. Following definitions 9.3, 9.4, and 9.5, of functions P,R and L in any space, it is observed that a "minor triad" in root position using property of matrix multiplication operate in same way.

$$W_P^{K^u_{TI}[a,b,c]} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ w+a \\ w+a+b \end{bmatrix} = \begin{bmatrix} w+a+b \\ w+b \\ w \end{bmatrix}$$

$$V_R^{K^u_{TI}[a,b,c]} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ w+a \\ w+a+b \end{bmatrix} = \begin{bmatrix} w+2a+b \\ w+a+b \\ w+a \end{bmatrix}$$

$$U_L^{K^u_{TI}[a,b,c]} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ w+a \\ w+a+b \end{bmatrix} = \begin{bmatrix} w+a \\ w \\ w-b \end{bmatrix}$$

Matrix U works as R for "major chords" (X, Y, Z) and V operates as L. Matrix representations drive the structural distances of triads (x, y, z) and (X, Y, Z), according to the PLR functions, regardless of the simplicial spaces in which the triads are found.

Definition 9.15. *(P^Ψ functions over minor triad (x,y,z) in K[a,b,c])*

$$P^\Psi(w, w+a, w+a+b) = (w+2a, w+a, w+a-b)$$

Theorem 9.16 (Harmonic Transformation P^Ψ_1)**.** *The function P^Ψ in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad, which is also equivalent to the $3x3$ square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$P^{\Psi\mathcal{K}^u_{TI}[a,b,c]}(x, y, z) = I_{2y}(x, y, z)^{\mathcal{K}^u_{TI}[a,b,c]} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\mathcal{K}^u_{TI}[a,b,c]}$$

Proof. Using the definition 9.1 and 9.15. First, inversion transform (x, y, z) in $(-w, -w-a, -w-a-b)$. Then, adds $2y = 2(w+a) = 2w+2a$ to each inverted element of the triad chord. This should satisfies the structural equality of P^Ψ function over (x, y, z):

$$\begin{aligned} -w + 2w + 2a &= w + 2a \\ -w - a + 2w + 2a &= w + a \\ -w - a - b + 2w + 2a &= w + a - b \end{aligned}$$

Also, proof of this equality on matrix representation must be demonstrated using matrix multiplication property

$$\begin{bmatrix} -1(w)+ & 2(w+a)+ & 0(w+a+b) \\ 0(w)+ & 1(w+a)+ & 0(w+a+b) \\ 0(w)+ & 2(w+a)+ & -1(w+a+b) \end{bmatrix} = \begin{bmatrix} w+2a \\ w+a \\ w+a-b \end{bmatrix}$$

This completes the proof and verifies the formula. □

The definition of the function R^Ψ maintains the element (x, y, z), where the tonic $x = w$ is a common note and becomes the fifth of the major chord Z. The third $y = w + a$ becomes the tonic of the major triad X, following the path $w + a - 2a - b = w - a - b$. Finally, the fifth z becomes the third Y of the major triad: $w + a + b - a - b - a = w - a$.

Definition 9.17. *(R^Ψ functions over minor triad (x,y,z) in K[a,b,c])*

$$R^\Psi(w, w+a, w+a+b) = (w, w-a, w-a-b)$$

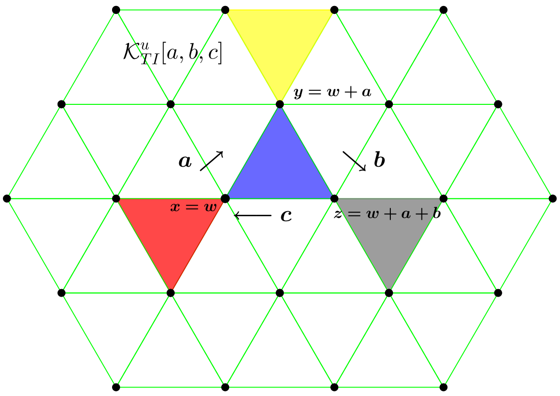


FIGURE 5. PLR^{Ψ} functions over simplicial complexes $\mathcal{K}^{u}_{TI}[a,b,c]$. P^{Ψ} in yellow, R^{Ψ} in red, L^{Ψ} in gray. The major base triad is plotted on the central blue equilateral triangle.

Theorem 9.18 (Harmonic Transformation R^{Ψ}_{1}). *The function R^{Ψ} in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad minus the double of distance a, which is also equivalent to the 3x3 square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$R^{\Psi\mathcal{K}^{u}_{TI}[a,b,c]}(x,y,z) = I_{2y-2a}(x,y,z)^{\mathcal{K}^{u}_{TI}[a,b,c]} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\mathcal{K}^{u}_{TI}[a,b,c]}$$

Proof. Using the definition 9.1 and 9.17. First, inversion transform (x, y, z) in $(-w, -w-a, -w-a-b)$. Then, adds $2y - 2a = 2(w + a) = 2w + 2a - 2a = 2w$ to each inverted element of the triad chord. This should satisfies the structural equality of R^{Ψ} function over (x, y, z):

$$\begin{array}{rcl} -w + 2w & = & w \\ -w - a + 2w & = & w - a \\ -w - a - b + 2w & = & w - a - b \end{array}$$

Also, proof of this equality on matrix representation must be demonstrated using matrix multiplication property

$$\begin{bmatrix} 1(w)+ & 0(w+a)+ & 0(w+a+b) \\ 2(w)+ & -1(w+a)+ & 0(w+a+b) \\ 2(w)+ & 0(w+a)+ & -1(w+a+b) \end{bmatrix} = \begin{bmatrix} w \\ w-a \\ w-a-b \end{bmatrix}$$

This completes the proof and verifies the formula. □

It can be seen that expression I_{2y-2a} is equal to the I_{2x} inversion of the table 1 that we show, but, the expression has been reformulated in terms of y, and for the specified support space $\mathcal{K}$. The following definition of L^{Ψ} is also based on the figure 5. We see that the third $y = w + a$ follows the path $+b - c$, so, $w + a + b + a + b = w + 2a + 2b$; that is, it becomes the fifth Z of the major triad. On the other hand, the root $x = w$, follows the path $w - c + b = w + a + b + b = w + a + 2b$. Moreover, the expression I_{2y+2b} is equal to the I_{2z} inversion.

Definition 9.19. *(L^{Ψ} functions over minor triad (x,y,z) in K[a,b,c])*

$$L^{\Psi}(w, w + a, w + a + b) = (w + 2a + 2b, w + a + 2b, w + a + b)$$

Theorem 9.20 (Harmonic Transformation L^{Ψ}_{1}). *The function L^{Ψ} in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad plus the double of distance b, which is also equivalent to the 3x3 square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$L^{\Psi\mathcal{K}^{u}_{TI}[a,b,c]}(x,y,z) = I_{2y+2b}(x,y,z)^{\mathcal{K}^{u}_{TI}[a,b,c]} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\mathcal{K}^{u}_{TI}[a,b,c]}$$

Proof. Using the definition 9.1 and 9.19. First, inversion transform (x, y, z) in $(-w, -w-a, -w-a-b)$. Then, adds $2y + 2b = 2(w + a) + 2b = 2w + 2a + 2b$ to each inverted element of the triad chord. This should satisfies the structural equality of L^Ψ function over (x, y, z):

$$\begin{array}{rcl} -w + 2w + 2a + 2b & = & w + 2a + 2b \\ -w - a + 2w + 2a + 2b & = & w + a + 2b \\ -w - a - b + 2w + 2a + 2b & = & w + a + b \end{array}$$

Also, proof of this equality on matrix representation must be demonstrated using matrix multiplication property

$$\begin{bmatrix} -1(w)+ & 0(w+a)+ & 2(w+a+b) \\ 0(w)+ & -1(w+a)+ & 2(w+a+b) \\ 0(w)+ & 0(w+a)+ & -1(w+a+b) \end{bmatrix} = \begin{bmatrix} w + 2a + 2b \\ w + a + 2b \\ w + a + b \end{bmatrix}$$

This completes the proof and verifies the formula. □

For major chords the following definitions are taken.

Definition 9.21. *(P^Ψ functions over major triad (X,Y,Z) in K[a,b,c])*

$$P^\Psi(w, w + b, w + a + b) = (2w + 2b, w + b, w - a + b)$$

Theorem 9.22 (Harmonic Transformation P_2^Ψ)**.** *The function P^Ψ in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third Y of the triad, which is also equivalent to the 3x3 square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$P^{\Psi\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^{\mathcal{K}^u_{TI}[a,b,c]}$$

Proof. Using definitions 9.2 and 9.21, develop a proof similar to the general theorem for the function P^Ψ on (x,y,z). □

Definition 9.23. *(R^Ψ functions over major triad (X,Y,Z) in K[a,b,c])*

$$R^\Psi(w, w + b, w + a + b) = (w + 2a + 2b, w + 2a + b, w + a + b)$$

Theorem 9.24 (Harmonic Transformation R_2^Ψ)**.** *The function R^Ψ in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad plus the double of distance a, which is also equivalent to the 3x3 square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$R^{\Psi\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y+2a}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^{\mathcal{K}^u_{TI}[a,b,c]}$$

Proof. Using definitions 9.2 and 9.23, develop a proof similar to the general theorem for the function R^Ψ on (x,y,z). □

Definition 9.25. *(L^Ψ functions over major triad (X,Y,Z) in K[a,b,c])*

$$L^\Psi(w, w + b, w + a + b) = (w, w - b, w - a - b)$$

Theorem 9.26 (Harmonic Transformation L_1^Ψ)**.** *The function L^Ψ in a simplicial complex $\mathcal{K}$ with interval distances a, b, c, such $a + b + c \equiv 0 \pmod{12}$ is equivalent to the inversion function I_n, such that n is the double of the third y of the triad minus the double of distance b, which is also equivalent to the 3x3 square matrix representation with integers coefficients over $\mathbb{Z}_{12}$. This is expressed through the formula:*

$$L^{\Psi\mathcal{K}^u_{TI}[a,b,c]}(X, Y, Z) = I_{2Y-2b}(X, Y, Z)^{\mathcal{K}^u_{TI}[a,b,c]} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^{\mathcal{K}^u_{TI}[a,b,c]}$$

Proof. Using definitions 9.2 and 9.25, develop a proof similar to the general theorem for the function L^Ψ on (x,y,z). □

10. Conclusions

In summary, respect to the first objetive we found another form of calculate the equivalence between PLR^{Ψ}, PLR and I_n multiplying by 2 only one component of the triad in root position. In relation with the second objetive, some matrices associated with the PLR^{Ψ} functions can be reduced to inversion operations among them, which also shows the notions of symmetry in the set of these matrices. Respect to third objetive, the "Harmonic Transformation Theorems" have been constructed in the chromatic system, but only changing the ring of the integers where they operate it can be use in diatonic system. The "Harmonic Transformations" demonstrates that the equivalence in any 2-simplicial complex of Neo-Riemannian functions PLR, PLR^{Ψ} and I_n can be quickly calculate using one element of the chord and operating some distances of the simplicial complex. The matrices representations of PLR^{Ψ} act equally in any simplicial space as well as the matrices U, V and W for PLR previously constructed in [FN18].

Future studies could fruitfully explore other symmetries by transpose or invert PLR^{Ψ} matrices. Which would lead to finding equivalences with composite PLR functions and other inversion symmetries I_n in a simplicial complex.

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